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Taxicab Geometry as a Vehicle for the Journey Toward Enlightenment

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In casual conversation, many (perhaps most) individuals are impatient with what they regard as slight distinctions of meaning. This impatience with fine-grained semantic sensitivity is reflected in the popularity of such pejorative expressions as “splitting hairs” and “just semantics.” The reigning attitude is that individuals who pay attention to apparently small differences in the definitions of words are pedantic and tedious. But slight differences in meaning can be surprisingly meaningful.

While scientific concepts, such as entropy or linkage disequilibrium are usually encountered only after years of schooling, distance is the sort of concept that everyone encounters before advancing very far along any path of study. A school age child has a sense of distance – how far from home to school or to a playmate’s house. Perhaps even a baby taking its first steps has some ability to gauge the distance from a gait-steadying piece of furniture to the outstretched arms of an encouraging parent. Nevertheless, such familiarity is no guarantee of comprehensive understanding.

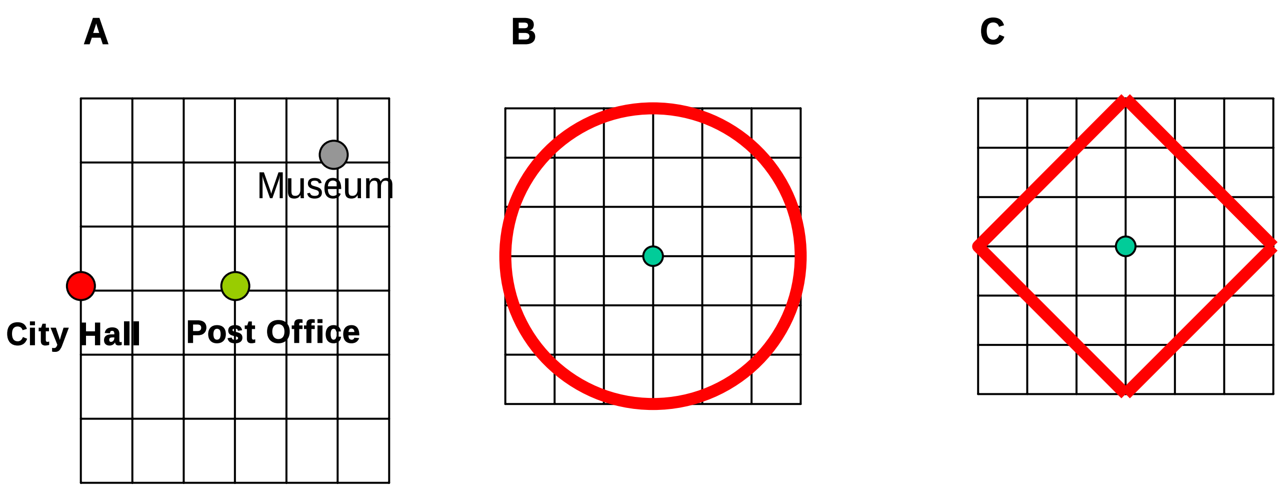

Figure 1. A) Geometric relationships between the Post Office, City Hall, and the Museum. B) Circle of radius 3 units based on Euclidean distance. C) Circle of radius 3 units based on taxicab distance. Figures adapted from: Krause, E.F. Taxicab Geometry: An Adventure in Non-Euclidean Geometry. Dover Publications, Inc., New York, 1975, 1986.

A cursory look at Figure 1A reveals three locations: the Post Office, City Hall, and the Museum. It would be regarded as routine to inquire whether City Hall or the Museum is closer to the Post Office. One might expect that such a simple question would readily elicit a clear, definitive response. If one relies on Euclidean geometry to provide the answer, then it is apparent that the Post Office is closer to the Museum than to City Hall. In Euclidean geometry, distance (d_E) between points A and B, is based on the Pythagorean Theorem:

$$d_E(A, B) = \sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2]}$$

where the X-axis and Y-axis coordinates of points A and B are, respectively, (a_1, a_2) and (b_1, b_2). By this definition, the Museum is $\sqrt{8}$ blocks from the Post Office, while City Hall is $\sqrt{9}$ blocks (3 blocks) from the Post Office and since $\sqrt{8} < \sqrt{9}$, the Museum is closer than City Hall to the Post Office.

However, one can conceive of using a measure of distance that differs from the Euclidean (i.e., as the crow flies) and is appropriate for the question at hand. After all, individuals using cars, buses, or their feet to get around do not travel as “the crow flies.” This non-

Euclidean distance can be referred to as taxicab distance (d_T), after Professor E. F. Krause (whose book on taxicab geometry provided my introduction to this corner of mathematics). Taxicab distance is defined for points A (a_1, a_2) and B (b_1, b_2) as:

$$d_T(A, B) = |a_1 - b_1| + |a_2 - b_2|$$

where $|a_1 - b_1|$ is the absolute value of the difference between a_1 and b_1.

This measure of distance corresponds to how a taxi driver might calculate the distances between buildings in a densely-settled urban environment.[1] By this measure, the Museum is 4 blocks away from the Post Office, while City Hall is only 3 blocks away. Thus, while the Museum is closer to the Post Office than is City Hall according to Euclidean distance, City Hall is closer if one uses taxicab distance. Professor Krause aptly suggests: "Taxicab geometry is a more useful model of urban geography than is Euclidean geometry. Only a pigeon would benefit from the knowledge that the Euclidean distance from the Post Office to the Museum is $\sqrt{8}$ blocks while the Euclidean distance from the Post Office to the City Hall is $\sqrt{9} = 3$ blocks. This information is worse than useless for a person who is constrained to travel along streets or sidewalks. For people, taxicab distance is "real" distance. It is *not* true, for people, that the Museum is "closer" to the Post Office than the City Hall is. In fact, just the opposite is true."

Such a rank reversal of proximity suggests that the innocent, and apparently straightforward, question about which building, City Hall or the Museum, is closer to the

Post Office, is actually not sufficiently precise to admit of a single answer. In order to attain such a degree of precision, a particular meaning must be attached to the term "distance." In other words, in this case, a mathematical formula must be specified. It seems likely that most people switch between different definitions of distance unconsciously. They are presumably not aware that when they are walking about an urban landscape with well-defined blocks, they are employing a non-Euclidean form of geometry to determine how far they must travel to get from one location to another.

All of the geometric implications of substituting taxicab distance for Euclidean distance may not be immediately apparent. Figures 1B and 1C present the sets of points that are equidistant from a central point (i.e., circles) based on, respectively, Euclidean distance and taxicab distance. What is striking about the comparison is that a circle based on taxicab distance is identical to what, in Euclidean geometry, we call a square. Thus, while taxicab geometry is identical to Euclidean geometry in all respects except for the definition of distance, the deductive consequences of that one 'simple' change are both far-reaching and dramatic. The broader lesson is that a relatively lengthy series of logical steps may be required to derive some of the logical consequences of a set of premises or axioms. In other words, what is deductively implicit can require substantial intellectual effort to be rendered explicit.

A further implication of the disparity between Euclidean and taxicab measures of distance arises from equating the distance between points with the similarity of the objects that can be represented by those points in some descriptive space. If any sorts of entities can be represented by ordered n-tuples of numbers in some n-dimensional coordinate system, then the similarities of those entities can be taken to increase as the distance between their corresponding points decreases and vice versa. Thus, the existence of more than one definition of distance implies that for entities described by two or more dimensions there is more than one way to assess their extents of similarity. Since assessment of similarity underlies all classification, the implication is that more than one reasonable classification is possible for these entities. The purpose of the classification may determine the most appropriate approach to classification, a notion that can be encapsulated in the Principle of Purpose-Dependent Ontology.

The implications of such classificatory flexibility for everyday life are pervasive. Consider the following riddle: What do the Bush-Gore election controversy of 2000, the implosion of Enron, and the Human Genome Project have in common? In each case, an absence of a precise (and prospective) definition led to confusion and or deception due to the failure of reproducible enumeration. In other words, it is exceedingly difficult to count, reliably or reproducibly, what you cannot define precisely.

Hence, the lack of clear prospective rules regarding the translation from punched chads to votes led to controversy over the outcome of the presidential election of 2000. An unrecognized, and in this case unwarranted, flexibility in the precise definition of profits allowed the executives at Enron, and many other public corporations, to mislead investors, clients, and others as to the financial positions of their respective companies. Finally, the leaders of the rival projects devoted to determining the complete nucleotide sequences of some putatively representative human genomes were unable to reach consensus on the number of human genes due to their failure to apply gene-counting algorithms that implicitly accepted the same definition of “gene.”

The same sorts of confusion can obtain even when enumeration is not the primary issue. Thus, there is a potentially endless list of terms and concepts that are open to various interpretations. Concepts, of relevance to all citizens, such as justice, fairness, liberal, conservative, patriot, war, liberty, happiness, merit, and rich can vary in meaning in ways that influence personal decisions and public policies

Experience with those who lack patience for fine distinctions in language has led me to the formulation of what can be referred to as The Bold Ontological Hypothesis: Most of the time, most people do not know (precisely) what they are talking about. The only way to know what you are talking about is to exert the extra effort it takes to know *exactly what* you are talking about.

Notes:

1. For the present purposes, we can limit ourselves, thereby simplifying the analysis, to distances between points on corners in a somewhat idealized urban environment laid out with two-way north-south and east-west streets. A more realistic treatment permitting calculation of distances between any pairs of points reinforces the main conclusions but at the price of accessibility and clarity.